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QUANTUM EFFECTS IN INFORMATION TRANSMISSION SYSTEMS

by I. A. Deryugin and V. N. Kurashov

Izdatel'stvo Naukova Dumka
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QUANTUM EFFECTS IN INFORMATION TRANSMISSION SYSTEMS

I. A. Deryugin, V. N. Kurashov

ABSTRACT. The basic assumptions of information theory change radically in the optical and infrared bands for a low level of eigen noise in the communication channel. The general expression for the entropy capacity of narrow-band and wide-band communication systems is obtained, and the requirements for optimum signal characteristics are determined.

High frequencies have been employed in an ever increasing amount during recent years in communication technology. The development of lasers and amplifiers has led to great success in utilizing the infrared and visible portion of the electromagnetic wave spectrum. There is no doubt that effective systems of communication in space and on the earth will be developed in the near future in this area. The high carrier frequency in conjunction with the very low noise level of these systems provides for an enormous information capacity of the communication channel. /292*

However, under these conditions different quantum phenomena which may be disregarded in normal radio technology are beginning to play an important role. In particular, definite changes must be implemented in the basic assumptions of the classical information theory: the model of a continuous communication channel, its information capacity, and the optimum spectral characteristics of the signal. Let us investigate a simple example. Employing the classical model of a continuous communication channel, Shannon (Ref. 1) demonstrated the fact that the transmissivity of a channel with a wide band W for an average-strength signal P_a in the presence of additive white noise P_n may be determined by the following formula

$$C = W \ln\left(1 + \frac{P_a}{P_n}\right). \tag{1}$$

Within the limits of customary noise temperatures, this expression coincides with our intuitive concepts, and provides a good description of the information properties of a communication channel. In the region where $T_n \to 0$ -- i.e., $P_n \to 0$ -- expression (1) strives to infinity in the case of arbitrary pass bands. This result is customarily explained by the fact that in the absence of noise any arbitrarily close signal levels may be distinguished, and consequently the number of initial symbols for the transmitted signal may be made infinitely large in the case of limited average strength P_a. As a result, the transmission of any one signal contains in itself an infinitely large amount of information. This line of reasoning, which is absolutely correct when viewed from classical physics, is in direct contradiction to the principle

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of uncertainty. In actuality, it is impossible to measure a field with arbitrary accuracy within a finite period of time, since the following relationship holds

$$\Delta E \Delta t \geqslant \hbar. \tag{2}$$

This relationship leads to the fact that, even when there is no noise, the information capacity of the channel remains limited. Thus, expression (1) no longer provides a correct description of a real communication system. /293

Certain changes in information theory appeared in different forms in the studies of different authors before the discovery of lasers. For example, Gabor (Ref. 2) introduced the concept of "quantization" as applied to communication channels. He was apparently the first one to consider the influence of quantum noise on the general noise properties of radiation receivers. The fundamental study by Callen and Welton (Ref. 3) made it possible to analyze the noise of amplifiers at very low noise temperatures. It was found that noise is limited from below by the so-called fluctuations of the zero field, whose strength equals $h\nu W$, where W is the instrument pass band.

The conclusion was immediately reached that the information capacity of information receivers remains limited even at absolute temperatures which strive to zero. Several authors have called attention to this fact [see, for example, (Ref. 15-11)]. However, these studies did not provide a comprehensive picture of the quantum phenomena in communication channels. Stern (Ref. 4, 5) and Gordon (Ref. 6, 8) developed the model of a quantum channel. On the basis of this model, they considered not only noise of the zero field, but also the phenomena related to the discrete nature of electromagnetic radiation. The application of this model to different communication systems revealed interesting new laws, which were not described by classical information theory even when zero field oscillations were taken into consideration. The discussion presented below primarily deals with the studies (Ref. 4-8). We shall also present certain original results pertaining to wide-band communication channels.

Model of a Quantum Communication Channel

The model of a photon channel may be formulated from the two basic postulates of quantum theory:

$$E = h\nu = \hbar\omega; \tag{3}$$

$$\Delta E \Delta t \geqslant \hbar. \tag{4}$$

The introduction of the first postulate indicates the change from a continuous (wave) description of the electromagnetic field to a discrete description (quantum). The second postulate imposes a limitation upon the number of independent variables which describe the radiation. In actuality, it is necessary to know the $2\Delta\nu$ variables in classical electrodynamics in order to determine the state of the field whose frequencies are concentrated in the $\Delta\nu$ band. For example, these may be the amplitudes and phases for expanding the field in Fourier series containing $\Delta\nu$ terms, or the values of the electric and magnetic field amplitudes of each type of orthogonal wave. In quantum mechanics, it follows from relationship (4) that the measurements of amplitude and phase, or of the magnetic and electric fields, are not independent. Only $\Delta\nu$ independent /294 measurements may be performed per second, and consequently the state of the field may be determined by the $\Delta\nu$ variables.

Let us now represent the quantum channel in the form of a portion of space (time -- frequency) which is delineated by the band $\Delta\nu$ and the time τ. Let us divide the selected portion of space into an element having the dimension $\Delta\nu\Delta t$. In order to satisfy the uncertainity principle, we must set

$$\Delta\nu\Delta t = 1. \tag{5}$$

Let us now assume that the signal source concentrates a certain number of photons in each element. Thus, each element represents a degree of freedom of the signal. It is assumed that the channel has a narrow band, in the sense that the photon frequencies in each element cannot be distinguished. Under these conditions, the role of the receiver consists of determining the number of photons in an element. Thus, the receiver represents an ideal quantum recorder. The physical feasibility of such a channel and its efficiency for different information receivers will be discussed below. We shall obtain the maximum entropy which the signal may have in such a channel. According to statistical mechanics, the entropy H per one degree of freedom may be written in the following form

$$H = -\sum_{n=0}^{\infty} p(n) \ln p(n), \tag{6}$$

where p(n) is the probability that the given element contains exactly n photons. The condition that the mean number of photons is retained

$$\sum_{n=0}^{\infty} np(n) = \bar{N}, \tag{7}$$

and also the normalization condition

$$\sum_{n=0}^{\infty} p(n) = 1. \tag{8}$$

are imposed on the distribution function p(n).

It may be readily seen that condition (7) is identical to the condition that the mean strength is retained for a channel having a narrow band. In actuality, multiplying by $h\nu\Delta\nu$, we obtain

$$\sum_{n=0}^{\infty} nh\nu p(n)\,\Delta\nu = \bar{N}h\nu\Delta\nu \equiv \bar{P}. \tag{9}$$

In order to determine the maximum possible entropy, we shall vary (6) in terms of the distribution function p(n) under the conditions (7) and (8). We then obtain the equation /295

$$-\ln p(n) - 1 + \lambda n + \mu = 0,$$

where λ and μ are arbitrary constants.

We thus have

$$p(n) = \alpha e^{\lambda n}, \quad \alpha = e^{\mu - 1}. \tag{10}$$

The constants α and λ may be determined from conditions (7) and (8):

$$\alpha \sum_0^\infty n e^{\lambda n} = \overline{N};$$

$$\alpha \sum_0^\infty e^{\lambda n} = 1.$$

Performing summation, we obtain the following system of equations

$$\left.\begin{aligned} \frac{\alpha}{1-e^\lambda} &= 1 \\ \frac{\alpha e^\lambda}{(1-e^\lambda)^2} &= \overline{N} \end{aligned}\right\}, \tag{11}$$

from which we have

$$\left.\begin{aligned} \lambda &= -\ln\left(1+\frac{1}{\overline{N}}\right) \\ \alpha &= \frac{1}{1+\overline{N}} \end{aligned}\right\}. \tag{12}$$

We finally have

$$p(n) = \frac{1}{1+\overline{N}}\left(\frac{\overline{N}}{1+\overline{N}}\right)^n. \tag{13}$$

We may find the maximum entropy of the photon distribution in the channel

$$\begin{aligned} H &= -\sum_{n=0}^\infty \alpha e^{\lambda n}\ln(\alpha e^{\lambda n}) = -\left(\frac{\alpha\ln\alpha}{1-e^\lambda}+\frac{\alpha\lambda e^\lambda}{(1-e^\lambda)^2}\right) = \\ &= -\ln\alpha - \lambda\overline{N} = \ln(1+\overline{N}) + \overline{N}\ln\left(1+\frac{1}{\overline{N}}\right). \end{aligned} \tag{14}$$

The rate at which entropy arrives, i.e., the entropy per unit of time, is

$$C = \frac{H}{\Delta t}, \tag{15}$$

or, with allowance for (5),

$$C = \Delta\nu H = \Delta\nu\ln(1+\overline{N}) + \Delta\nu\overline{N}\ln\left(1+\frac{1}{\overline{N}}\right). \tag{16}$$

Employing relationship (9), we obtain

$$C = \Delta\nu\ln\left(1+\frac{P}{h\nu\Delta\nu}\right) + \frac{P}{h\nu}\ln\left(1+\frac{h\nu\Delta\nu}{P}\right). \tag{17}$$

The last formula is a quantum analogue of expression (1) in the case /296
$P_n \to 0$. The first term in the right hand side of (17) coincides with the right hand side of (1), if we take the fact into account that the noise is limited by oscillations of the zero field. The second term of (17) is a purely quantum term caused by the discrete nature of the photon field. Under ordinary conditions, the first term plays the basic role. If the mean number of photons per second is less than unity, the second term dominates.

We should point out that it is not possible to describe a channel by specifying the precise number of photons at each moment of time. However, the number of degrees of freedom for the signal does not depend on the particular method of description. Consequently, the expressions which we obtained must be valid when any other variables are selected, as it may be assumed that expression (17) is universally valid.

In conclusion, we may find the rate at which information is transmitted over a quantum channel having narrow band when there is additive noise. The amount of information transmitted under these conditions cannot be greater than the difference between the total entropy of the signal and the noise, and the entropy of only the noise. In order that this value may reach a maximum, the signal and the noise -- taken together -- must have the statistical properties of the noise. Thus, the rate at which information is transmitted may be determined by the expression

$$C = \Delta\nu\,(H_{P_a+P_n} - H_{P_n}) = \Delta\nu \ln\left(1 + \frac{P_a + P_n}{h\nu\Delta\nu}\right) + \frac{P_a + P_n}{h\nu}\ln\left(1 + \frac{h\nu\Delta\nu}{P_a + P_n}\right) - \Delta\nu\ln\left(1 + \frac{P_n}{h\nu\Delta\nu}\right) - \frac{P_n}{h\nu}\ln\left(1 + \frac{h\nu\Delta\nu}{P_n}\right),$$

or

$$C = \Delta\nu\ln\left(1 + \frac{P_a}{P_n + h\nu\Delta\nu}\right) + \frac{P_a + P_n}{h\nu}\ln\left(1 + \frac{h\nu\Delta\nu}{P_a + P_n}\right) - \frac{P_n}{h\nu}\ln\left(1 + \frac{h\nu\Delta\nu}{P_n}\right). \quad (18)$$

It is of significant interest to determine the degree to which the limiting value (18) of the information transmittal rate may be attained. In addition to the customary conditions of optimum encoding, certain interesting problems arise which are related to the possibility of information being transmitted by a signal. We shall deal with these problems below, when discussing the quantum recorder.

Channel without Noise with an Infinite Pass Band

Let us investigate a signal whose frequencies cannot be assumed to be identical within an accuracy of the uncertainity principle, essentially the photon distribution with respect to frequency. This case is realized, for example, in a wide-band channel when a modulated signal is being transmitted. A similar /297
problem arises when we investigate a system of multi-channel communication with frequency separation of the channels. In both cases, it is interesting to find the optimum (with respect to information theory) strength distribution in terms of frequency.

The study (Ref. 4) investigates a similar problem for a channel without noise with an infinite pass band. Let us investigate it in greater detail. We shall assume, as was done in (Ref. 4), that the source generates a large number of narrow lines independently. The photon energies of each such line are approximately the same. The mean quantum occupation number of the i^{th} line will be designated by N_i. As was indicated previously, the entropy of such a narrow-band signal is limited by the quantity

$$H_i = N_i \ln\left(1 + \frac{1}{N_i}\right) + \ln(1 + N_i). \quad (19)$$

The total entropy of the source signal per unit of time equals

$$H_t = \frac{1}{\Delta t} \sum_{i=1}^{\infty} H_i. \tag{20}$$

In order to satisfy the uncertainity principle, we must set the following, just as previously,

$$\Delta t \Delta \nu = 1.$$

We then have

$$H = \Delta \nu \sum_{i=1}^{\infty} H_i = \Delta \nu \sum_{i=1}^{\infty} [N_i \ln\left(1 + \frac{1}{N_i}\right) + \ln(1 + N_i)]. \tag{21}$$

Investigating the extremum of this value, we obtain the dependence $N_i = N_i(\nu)$, at which (21) assumes a maximum value. Let us impose an additional condition that the average strength be constant

$$\Delta \nu \sum_{i=1}^{\infty} N_i h \nu_i = P. \tag{22}$$

The variational problem for the conditional extremum (21) leads to the following equation

$$\frac{\partial}{\partial N_i} \left\{ \Delta \nu \sum_{i=1}^{\infty} \left[N_i \ln\left(1 + \frac{1}{N_i}\right) + \ln(1 + N_i) \right] + \lambda \Delta \nu \sum_{i=1}^{\infty} N_i h \nu_i \right\} = 0.$$

We thus have

$$\left. \begin{aligned} &\ln\left(1 + \frac{1}{N_i}\right) + \lambda h \nu_i = 0 \\ &N_i = \frac{1}{e^{-h\nu_i \lambda} - 1} \end{aligned} \right\}. \tag{23}$$

In order to find λ, we substitute (23) in (22): /298

$$\Delta \nu \sum_{i=1}^{\infty} \frac{h \nu_i}{e^{-\lambda h \nu_i} - 1} = P.$$

It is apparent that $\lambda < 0$, since in the opposite case the series diverges at infinity. Therefore, we shall replace λ by $-\lambda$ below, so that

$$\Delta \nu \sum_{i=1}^{\infty} \frac{h \nu_i}{e^{\lambda h \nu_i} - 1} = P; \tag{24}$$

$$N_i = \frac{1}{e^{\lambda h \nu_i} - 1}. \tag{25}$$

The quantity $\Delta\nu$ in this equation may be decreased until the condition $\Delta\nu\Delta t = 1$ is satisfied. We shall assume that $\Delta\nu$ may be made very small. Let us calculate λ for the case $\Delta\nu \to 0$. Then the sum may be replaced by the integral

$$\lim_{\Delta \nu \to 0} \sum_{i=1}^{\infty} \frac{h \nu_i \Delta \nu}{e^{\lambda h \nu_i} - 1} = \int_0^{\infty} \frac{h \nu d\nu}{e^{h \nu \lambda} - 1}.$$

The integral in the right hand side may be reduced to the following form

$$\frac{1}{h \lambda^2} \int_0^{\infty} \frac{x dx}{e^x - 1} = \frac{1}{h \lambda^2} \cdot \frac{\pi^2}{6} = P.$$

We thus determine the following

$$\lambda = \frac{\pi}{\sqrt{6Ph}}; \tag{26}$$

$$N(\nu) = \frac{1}{e^{\frac{h\nu\pi}{\sqrt{6\pi h}}} - 1}. \tag{27}$$

Let us introduce a certain effective temperature

$$T_{\mathrm{eff}} = \frac{1}{k\pi}\sqrt{6Ph}. \tag{28}$$

We then have

$$N(\nu) = \frac{1}{e^{\frac{h\nu}{kT_{\mathrm{eff}}}} - 1}. \tag{29}$$

Thus, the entropy of a quantum signal of an infinite band has a maximum in the case of the occupation number distribution (29). This is the Bose-Einstein distribution for a body which is heated to a temperature of T_{eff}. Introducing the spectral density of the radiation strength

$$P(\nu)\,d\nu = h\nu N(\nu)\,d\nu, \tag{30}$$

we obtain /299

$$P(\nu) = \frac{h\nu}{e^{\frac{h\nu}{kT_{\mathrm{eff}}}} - 1}. \tag{31}$$

Formulas (29) and (31) represent a natural generalization of classical theory, according to which the entropy is at a maximum in the case of a uniform spectral distribution of radiation emitted by an absolutely black body which is heated to the temperature T_{eff}:

$$P_{\mathrm{class}}(\nu) = kT_{\mathrm{eff}}. \tag{32}$$

In the quantum case, the optimum frequency distribution is also determined by the radiation of an absolutely black body. The signal ceases to be "white", i.e., its spectral density depends on frequency.

Expressions (28) and (29) make it possible to establish the fundamental limit for entropy which may be contained in a signal having the mean strength P. In actuality, for $\Delta\nu \to 0$ we have

$$H_{\max} = \int_0^\infty H(\nu)\,d\nu,$$

where

$$H(\nu) = N(\nu)\ln\left[1 + \frac{1}{N(\nu)}\right] + \ln[1 + N(\nu)] =$$

$$= \frac{1}{\exp\frac{h\nu}{kT_{\mathrm{eff}}} - 1}\ln\left[\exp\frac{h\nu}{kT_{\mathrm{eff}}}\right] + \ln\left[1 + \frac{1}{\exp\frac{h\nu}{kT_{\mathrm{eff}}} - 1}\right]. \tag{33}$$

Transforming this expression, we obtain

$$H_{\max} = \int_0^\infty \frac{h\nu\,d\nu}{kT_{\mathrm{eff}}\left[\exp\frac{h\nu}{kT_{\mathrm{eff}}} - 1\right]} + \int_0^\infty \ln\left[\frac{\exp\frac{h\nu}{kT_{\mathrm{eff}}}}{\exp\frac{h\nu}{kT_{\mathrm{eff}}} - 1}\right]d\nu. \tag{34}$$

After simple transformations, both integrals are reduced to the tabulated integral

$$\int_0^\infty \frac{xdx}{e^x - 1} = \frac{\pi^2}{6}.$$

We then have

$$H_{max} = \frac{2kT_{eff}}{h} \cdot \frac{\pi^2}{6}, \tag{35}$$

or, taking (28) into account, we have

$$H_{max} = \frac{1}{3} \pi \sqrt{\frac{6P}{h}}. \tag{36}$$

Such information which is maximally possible may be transmitted per unit of time over a channel with an infinite band without noise by a signal having the strength P. It is interesting to note that Gordon (Ref. 6) obtained such /300
an expression with a coefficient which was somewhat too low by direct determination of the number of possible distinguishable levels allowed by the uncertainity principle.

Channel without Noise with a Limited Pass Band

The problem examined above is of purely theoretical interest, since it is impossible to produce a communication system with an infinite pass band. It is much more interesting to determine the maximum possible rate at which information may be transmitted over a channel with a limited band. We shall now examine such a channel.

We should first note that the very method by which the preceding problem is solved indicates that, for a signal having a limited band, the entropy reaches a maximum for a distribution of the occupation numbers which also obeys the following law

$$N(\nu) = \frac{1}{\exp \frac{h\nu}{kT} - 1}.$$

This is due to the fact that the summation limits, which depend on the pass band, have no influence anywhere on the distribution calculation. The summation limits will be of importance when the temperature T is calculated, which in this case will not equal T_{eff}. Generally speaking, T depends both on the signal strength and on the pass band, and the stronger it is, the greater is the strength and the narrower is the pass band.

However, let us now examine the information transmission rate of a signal with the Bose-Einstein distribution, whose temperature does not depend on the pass band. When it is advantageous, we shall take this dependence into account. We shall everywhere assume that $\Delta\nu \to 0$, since we shall employ integrals, instead of sums.

The entropy of a signal having the Bose-Einstein distribution in a limited pass band W equals

$$H=\int_0^W H(\nu)\,d\nu=\int_0^W \frac{\frac{h\nu}{kT}}{\exp\frac{h\nu}{kT}-1}\,d\nu+\int_0^W \ln\left[\frac{\exp\frac{h\nu}{kT}}{\exp\frac{h\nu}{kT}-1}\right]d\nu. \tag{37}$$

In order to find H, let us introduce the substitution in the first integral

$$x=\frac{h\nu}{kT}$$

and the substitution in the second integral /301

$$y=\ln\left[\frac{\exp\frac{h\nu}{kT}}{\exp\frac{h\nu}{kT}-1}\right].$$

After simple transformations, we then obtain

$$H=\frac{kT}{h}\int_0^{\frac{hW}{kT}}\frac{xdx}{e^x-1}-\frac{kT}{h}\int_{\infty}^{\ln\left[\frac{\exp\frac{hW}{kT}}{\exp\frac{hW}{kT}-1}\right]}\frac{ydy}{e^y-1}.$$

Changing the integration limits in the second integral and the notation for the integration variable, we obtain

$$H=\frac{kT}{h}\left[\int_0^{\frac{hW}{kT}}\frac{xdx}{e^x-1}+\int_{\ln\left[\frac{\exp\frac{hW}{kT}}{\exp\frac{hW}{kT}-1}\right]}^{\infty}\frac{xdx}{e^x-1}\right].$$

Let us combine both integrals

$$H=\frac{kT}{h}\left[\int_0^{\infty}\frac{xdx}{e^x-1}+\int_a^b\frac{xdx}{e^x-1}\right], \tag{38}$$

where

$$\left.\begin{aligned} a&=\ln\left[\frac{\exp\frac{hW}{kT}}{\exp\frac{hW}{kT}-1}\right]\\ b&=\frac{hW}{kT}\end{aligned}\right\}. \tag{39}$$

The first of these integrals is a tabulated integral which equals $\frac{\pi^2}{6}$. Thus, the problem consists of calculating the second integral

$$I=\int_a^b\frac{xdx}{e^x-1}. \tag{40}$$

This integral cannot be expressed in terms of elementary functions. For this study, let us represent the integrand in the form of the series

$$I=\int_a^b\sum_{n=1}^{\infty}xe^{-nx}dx.$$

Changing the locations of summation and integration signs, we obtain /302

$$I=-\sum_{n=1}^{\infty}\left[\frac{xe^{-nx}}{n}+\frac{e^{-nx}}{n^2}\right]_a^b.$$

The first sum may be readily calculated

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}=-\ln(1-e^{-x}),\quad x>0.$$

We thus find

$$I=-a\ln(1-e^{-a})+b\ln(1-e^{-b})-\sum_{n=1}^{\infty}\frac{e^{-xn}}{n^2}\Big|_a^b.$$

Substituting a and b, we obtain

$$b\ln(1-e^{-b})-a\ln(1-e^{-a})=$$

$$=\frac{hW}{kT}\ln(1-e^{-\frac{hW}{kT}})-\frac{hW}{kT}\ln e^{-\frac{hW}{kT}}-\frac{hW}{kT}\ln(e^{\frac{hW}{kT}}-1)=0.$$

We finally have

$$H=\frac{kT}{h}\left[\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{e^{-nx}}{h^2}\Big|_a^b\right];$$

$$H=\frac{kT}{h}\left[\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{(1-e^{-\frac{hW}{kT}})^n-e^{-\frac{hW}{kT}n}}{h^2}\right]. \tag{41}$$

In the case $\frac{hW}{kT}\to 0$, the first sum vanishes, and the second sum strives to the value $\frac{\pi^2}{6}$ and $H\to 0$. In the case $\frac{hW}{kT}\to\infty$, the second sum equals zero, and the first sum equals $\frac{\pi^2}{6}$. We then have

$$H\to\frac{kT}{h}\cdot\frac{\pi^2}{3}=H_{max}.$$

If $\frac{hW}{kT}=\ln 2$, then both sums are equal, and we consequently have

$$H=\frac{kT\pi^2}{6}=\frac{H_{max}}{2}.$$

In order to study H for different W and T, let us find $\frac{\partial H}{\partial W}$ and $\frac{\partial H}{\partial T}$, first assuming that W and T are independent variables. This is valid for large pass /303 bands or for small strengths:

$$\frac{\partial H}{\partial W}=\frac{kT}{h}\sum_{n=1}^{\infty}\left[\frac{(1-e^{-\frac{hW}{kT}})^{n-1}e^{-\frac{hW}{kT}}}{n}+\frac{e^{-\frac{hW}{kT}}}{n}\right]\frac{h}{kT}= \tag{42}$$

$$=\frac{hW}{kT\left[\exp\frac{hW}{kT}-1\right]}-\ln\left[1-\exp\left(-\frac{hW}{kT}\right)\right]; \tag{43}$$

$$\frac{\partial H}{\partial T}=\frac{H}{T}-\frac{W}{T}\cdot\frac{\partial H}{\partial W}.$$

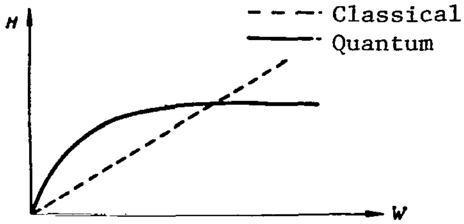

Figure 1

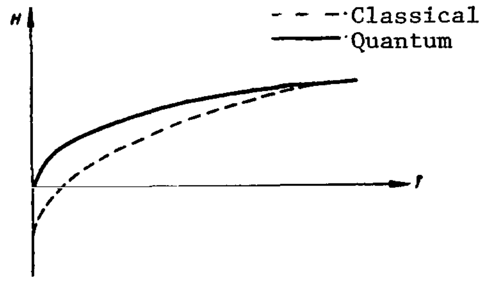

Figure 2

Let us find the asymptotic behavior of these quantities. First of all, the investigation shows that

$$\frac{\partial H}{\partial W} > 0 \text{ and } \frac{\partial H}{\partial T} > 0,$$

from which it follows that the entropy increases with an increase in the strength and the pass band.

Expanding the uncertain terms in formulas (42) and (43), we readily find that

$$\lim_{W\to 0}\frac{\partial H}{\partial W} = \infty;$$

$$\lim_{W\to\infty}\frac{\partial H}{\partial W} = 0\left(\frac{hW}{kT}e^{-\frac{hW}{kT}}\right);$$

$$\lim_{T\to 0}\frac{\partial H}{\partial T} = \frac{\pi^2 k}{3h}.$$

In order to find $\lim\limits_{T\to\infty}\frac{\partial H}{\partial T}$, let us calculate H approximately in the case $hW \ll kT$. We may then rewrite H in the following form

$$H = \int_0^W \frac{\frac{h\nu}{kT}}{\frac{h\nu}{kT}}d\nu + \int_0^W \ln\left[1+\frac{kT}{h\nu}\right]d\nu.$$

After simple calculations, we obtain

$$H = W + W\ln\left(1+\frac{kT}{hW}\right) + \frac{kT}{h}\ln\left(1+\frac{hW}{kT}\right). \tag{44}$$

We thus have

$$\left.\frac{\partial H}{\partial T}\right|_{hW \ll kT} = \frac{k}{h}\ln\left(1+\frac{hW}{kT}\right);$$

$$\lim_{T\to\infty}\frac{\partial H}{\partial T} = 0\left(\frac{hW}{kT}\right).$$

All the results obtained differ from the classical results. As is known: /304

$$H_{\text{class}} = W\ln\frac{P}{P_0}; \tag{45}$$

$$\frac{\partial H_{\text{class}}}{\partial W} = \ln\frac{P}{P_0} = \text{const}\,(P); \tag{46}$$

$$\frac{\partial H_{\text{class}}}{\partial P} \sim \frac{\partial H_{\text{class}}}{\partial T} = \frac{WP_0}{P}. \tag{47}$$

The results of these considerations are illustrated by graphs given in Figures 1-4.

/305

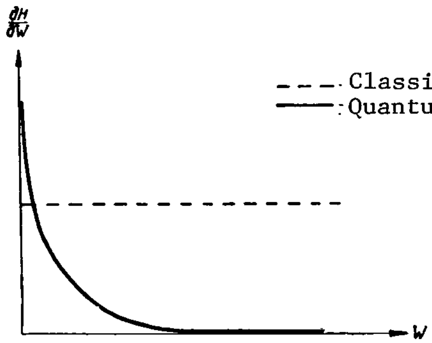

Figure 3

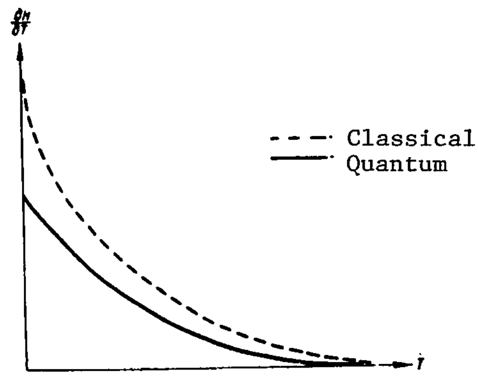

Figure 4

It may be seen from the above relationships that, in contrast to the classical formula, $\frac{\partial H}{\partial W}$ decreases with an increase in W approximately according to an exponential law. Thus, the widening of the channel band in the case hW < kT barely leads to an increase in the entropy.

The second interesting consequence is that for small signal strengths, the entropy increases when the strength increases much more slowly than would follow from classical theory. The following conclusion thus must be drawn: in the case hW < kT, in order to increase the amount of information which may be introduced in the signal it is valid to increase the pass band. On the other hand, in the case hW > kT, it is valid to increase the signal strength (since for $\frac{hW}{kT} \to \infty$ $\frac{\partial H}{\partial T}$ strives to zero only as $\frac{1}{T}$, while $\frac{\partial H}{\partial W}$ strives to zero as $e^{-\frac{hW}{kT}}$).

Let us now consider the dependence of the constant T on the pass band W. It is apparent that this dependence will be significant primarily for small pass bands or for large strengths, i.e., for the case $\frac{hW}{kT} \ll 1$. In actuality, T may be determined from the condition

$$\int_0^W \frac{h\nu d\nu}{\exp\frac{h\nu}{kT} - 1} = P_a$$

or

$$\frac{k^2T^2}{h}\int_0^{\frac{hW}{kT}}\frac{xdx}{e^x-1}=P_a$$

For large $\frac{hW}{kT}$, the integral differs very little from $\frac{\pi^2}{6}$, and therefore $T \sim \sqrt{P_a}$. For small $\frac{hW}{kT}$, the integral depends strongly upon $\frac{hW}{kT}$, and therefore we cannot assume that $T = \sqrt{P\ \text{const.}}$. For this case, assuming that $x \ll 1$, let us expand the integrand, confining ourselves to only the first term of the expansion: /306

$$\frac{k^2T^2}{h}\int_0^{\frac{hW}{kT}}\frac{xdx}{1+x-1}=kTW=P_a \tag{48}$$

Let us calculate the entropy for this case. It was shown above that for $hW \ll kT$ we have

$$H=W+W\ln\left(1+\frac{kT}{hW}\right)+\frac{kT}{h}\ln\left(1+\frac{hW}{kT}\right).$$

Substituting T, we obtain

$$H=W+W\ln\left(1+\frac{P_a}{hW^2}\right)+\frac{P_a}{hW}\ln\left(1+\frac{hW^2}{P_a}\right). \tag{49}$$

If we pass to the limit, formally letting $h \to 0$, we obtain

$$H=W\ln\frac{e^2P_a}{hW^2}, \tag{50}$$

which corresponds to the classical information theory in the following case

$$P_0=\frac{hW^2}{e^2}. \tag{51}$$

Expressions (50) and (51) determine the transition to the classical case. If it is assumed that the constant P_0 is arbitrary in the Shannon theory, and consequently the level from which the entropy is measured is also aribtrary, the quantum mechanical conclusion establishes the absolute measurement level which is determined by the strength of the quantum noise hW^2.

Channel with Noise

Let us assume that the channel contains thermoadditive noise with a specific strength distribution $P_n(\nu)$. We shall investigate what spectral distribution $P_a(\nu)$ the signal must have in order that it will transmit a maximum amount of information. In the classical case, this condition may be written in the following form (Figure 5).

$$P_a(\nu)+P_n(\nu)=\text{const.} \tag{52}$$

The signal must be large at those frequencies where the noise is small.

In order to determine the corresponding condition in the quantum case, let us write the information transmission rate when there is noise in the

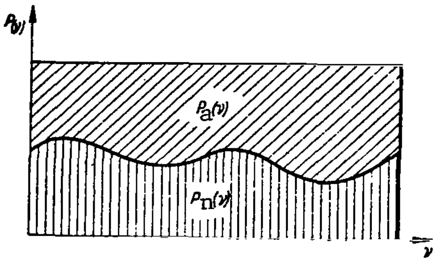

Figure 5

following form /307

$$C = \frac{1}{\Delta t}[H_{c+n} - H_n] =$$

$$= \sum_{i=1}^{\frac{W}{\Delta\nu}} \left\{ N_{c+n}(\nu_i) \ln\left[1 + \frac{1}{N_{c+n}(\nu_i)}\right] + \ln[1 + N_{c+n}(\nu_i)] - \right.$$

$$\left. - N_n(\nu_i) \ln\left[1 + \frac{1}{N_n(\nu_i)}\right] - \ln[1 + N_n(\nu_i)]\right\}. \quad (53)$$

The problem consists of trying to determine the extremum of this expression. This entails certain difficulties, in view of the fact that the exponential distribution is not additive, and consequently

$$N_{c+n} \neq N_c + N_n.$$

However, we shall employ the approximate equation $N_{c+n} = N_c + N_n$ which, generally speaking, is practically satisfied in real channels (Ref. 6).

Under the given assumption, the information transmission rate may be written as follows:

$$C = \sum_{i=1}^{\frac{W}{\Delta\nu}} \Delta\nu \left\{ [N_c(\nu_i) + N_n(\nu_i)] \ln\left[1 + \frac{1}{N_c(\nu_i) + N_n(\nu_i)}\right] + \right.$$
$$+ \ln[1 + N_c(\nu_i) + N_n(\nu_i)] - N_n(\nu_i) \ln\left[1 + \frac{1}{N_n(\nu_i)}\right] - \quad (54)$$
$$\left. - \ln[1 + N_n(\nu_i)]\right\}.$$

Let us find the extremum of this expression under the condition that total signal strength is constant

$$\sum_{i=1}^{\frac{W}{\Delta\nu}} N_c(\nu_i) h\nu_i \Delta\nu = \text{const}. \quad (55)$$

This problem leads to the equation

$$\frac{\partial C}{\partial N_c(\nu_i)} + \lambda \frac{\partial}{\partial N_c(\nu_i)} \sum_{i=1}^{\frac{W}{\Delta\nu}} N_c(\nu_i) h\nu_i \Delta\nu = 0.$$

Substituting C, we obtain

$$\ln\left[1 + \frac{1}{N_c + N_n}\right] + \lambda h\nu = 0, \quad (56)$$

or /308

$$N_c + N_n = \frac{1}{e^{-\lambda h\nu} - 1}.$$

The constant λ may be calculated from condition (55). It is apparent that $\lambda < 0$ based on the same considerations as in the preceding cases.

Thus, the maximum information transmission rate is achieved for given noise if the total ensemble produces the Bose-Einstein distribution (Figure 6):

$$P_a(\nu) + P_a(\nu) = \frac{h\nu}{e^{h\nu|\lambda|} - 1}. \tag{57}$$

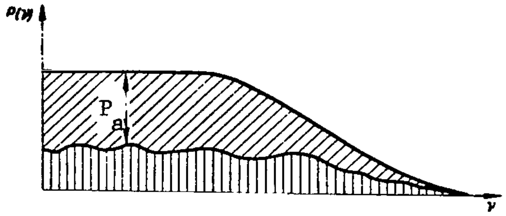

Figure 6

The constant in the right hand side of (55) must be selected so that the level of the signal strength everywhere exceeds the level of the noise strength; otherwise negative strengths will be encountered.

The result (57) differs from the classical result at high frequencies, where the signal strength must decrease. For low frequencies, for which $\frac{1}{|\lambda|} \gg h\nu$, we may write

$$P_a(\nu) + P_a(\nu) = \frac{h\nu}{1 + h\nu|\lambda| - 1} = \frac{1}{|\lambda|},$$

which coincides with (52).

We may readily find the maximum information transmission rate by this channel. Let us assume that the noise, and also the signal + noise, obey the Bose-Einstein distribution. For the case hW << kT, employing (49), we then obtain

$$\begin{aligned} C = H_{c+n} - H_n = \\ = W \ln\left(1 + \frac{P_a + P_n}{hW^2}\right) + \frac{P_a + P_n}{hW} \ln\left(1 + \frac{hW^2}{P_a + P_n}\right) - \\ - W \ln\left(1 + \frac{P_n}{hW^2}\right) - \frac{P_n}{hW} \ln\left(1 + \frac{hW}{P_n}\right) = \\ = W \ln\left(1 + \frac{P_a}{P_n + hW^2}\right) + \frac{P_a + P_n}{hW} \ln\left(1 + \frac{hW^2}{P_a + P_n}\right) - \\ - \frac{P_n}{hW} \ln\left(1 + \frac{hW^2}{P_n}\right). \end{aligned} \tag{58}$$

This formula is valid only in the case P_a, $P_n \gg hW^2$. If this inequality is not satisfied, we must employ expression (41).

Quantum Recorder /309

Let us first study different quantum systems, and let us make certain general remarks. Formulas were presented above which established the upper limit for the amount of information which may be contained in a signal. However, it is not entirely apparent that all of this information may be extracted from the signal at the receiving end of the channel. The essential difference from the classical case lies in the fact that in quantum mechanics the properties of the recording device inevitably influence the signal being recorded, and change it to a certain degree. Thus, we cannot draw any conclusion regarding the ability to achieve these conditions from the expressions obtained above. Under these conditions, the construction of an applicable receiving device acquires particular importance. If it is sufficient to specify the resulting receiver noise level in classical theory in order to describe the information properties, in the case of a quantum channel -- as we shall see below -- a special examination must be made for each type of receiving device, independently of its noise properties.

As has already been pointed out, a quantum recorder represents an ideal receiver when the model described for the quantum channel is employed. However, the very concept of "idealness" necessitates an additional discussion. We shall use the term mathematically ideal to describe an amplifier at whose output precisely Gn photons appear when n photons enter at the input, where G is the amplification coefficient. The information efficiency of this amplifier equals unity. As we shall see below, in physical terms the existence of a mathematically ideal amplifier is impossible. Each real amplifier only approximates this idealization to a certain extent. Therefore, we shall introduce the concept of a physically ideal amplifier, where the term "idealness" is employed in the sense of the absence of eigen noise. The investigation of such a amplifier considerably decreases the computational difficulties.

For small amounts of photons entering the input, it may be assumed that the device advanced by Weber is physically ideal. For large occupation numbers, a wave guide maser represents such a device in the case $T = 0°K$. Let us examine both of these receivers, and let us determine the extent to which they approximate a mathematically ideal receiver.

Binary recorder. If the mean occupation numbers of the photons are small, so that $P_a \ll h\nu W$, only two probabilities will differ from zero: for production of no photon, or for production of one photon. Let us assume that the receiver is distinguished by only two conditions: there are no photons at the input, or there is any number of photons at the input. It is apparent that this system operates efficiently only when there are small occupation numbers. We shall call this type of a device a binary quantum recorder.

Let us assume that the signal being transmitted consists of an arbitrary distribution of impulses, each of which has the duration W^{-1}. The probability for an impulse to be transmitted in any given time interval equals Q. If the receiver detects even one photon in this time interval, it records 1, and if no photon is detected it records 0. We shall assume that there is no noise in the channel. If an impulse is transmitted, the receiver may record both 1 and 0. If 0 is transmitted, the output signal always corresponds to 0. Figure 7 shows an illustration of the possible cases. If the signal is sufficiently weak when passing through the channel, the photon distribution at the input may be described by the Poisson law with a high degree of accuracy. According to this law, the probability p(n) of obtaining precisely n photons equals /310

$$p(n) = \frac{s^n}{n!} e^{-s}. \tag{59}$$

Thus, the probability of obtaining not even one photon equals e^{-s}, and the probability of obtaining one photon equals $1 - e^{-s}$. These probabilities are also shown in Figure 7.

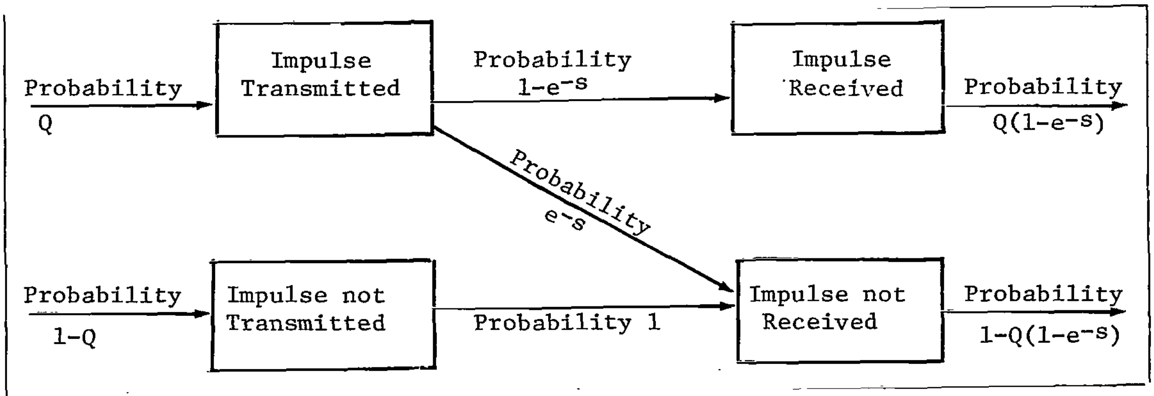

Figure 7

Let us determine the information efficiency of a receiver as follows:

$$\eta = 1 - \frac{H_x(y)}{H(y)}. \tag{60}$$

Here H (y) is the entropy of one symbol of information to be received which is determined by summation over the probability of all possible symbols; H_x (y) -- arbitrary entropy determined by the probability of receiving the symbol y when the symbol x enters the input:

$$H(y) = -\sum_y p(y) \ln p(y); \tag{61}$$

$$H_x(y) = -\sum_x p(x) \sum_y p_x(y) \ln p_x(y). \tag{62}$$

According to the well known Shannon theorem, the quantity

$$I = H(y) - H_x(y) \tag{63}$$

determines the amount of information extracted from the signal during reception, /311
and consequently η represents a measure of the receiver efficiency. If η = 1, then all the information contained in the signal is received. If η < 1, then a portion of the information is lost.

We may readily find H(y) and H_x(y) by employing (61) and (62). We have

$$P(1) = Q(1 - e^{-s});$$
$$P(0) = 1 - Q(1 - e^{-s});$$
$$P_1(0) = e^{-s};$$
$$P_1(1) = 1 - e^{-s};$$
$$P_0(0) = 1;$$
$$P_0(1) = 0.$$

Thus, performing summation, we obtain

$$H(y) = -Q(1 - e^{-s}) \ln [Q(1 - e^{-s})] - \\ -[1 - Q(1 - e^{-s})] \ln [1 - Q(1 - e^{-s})]; \tag{64}$$

$$H_x(y) = -Q[e^{-s} \ln e^{-s} + (1 - e^{-s}) \ln (1 - e^{-s})]; \tag{65}$$

$$I = -Q(1-e^{-s})\ln Q - [1 - Q(1-e^{-s})]\ln[1 - Q(1-e^{-s})] + + Qe^{-s}\ln e^{-s}; \quad (66)$$

$$\eta = 1 - \frac{Q(1-e^{-s})\ln[Q(1-e^{-s})] + [1 - Q(1-e^{-s})]\ln[1 - Q(1-e^{-s})]}{Q[e^{-s}\ln e^{-s} + (1-e^{-s})\ln(1-e^{-s})]}. \quad (67)$$

We should note that the quantities s and Q may be interrelated, if we take the fact into account that the mean number of photons entering the input in the period of time W^{-1} is maintained. Since the mean number of photons received in one impulse equals s, during the time W^{-1}, Qs photons are received on the average. Consequently, we have

$$Qs = \bar{N} = \text{const}. \quad (68)$$

Substituting Q from equation (68) in (66), we find the value of I for different $\bar{N}$ and s. Differentiating the equation obtained with respect to s and setting the derivative equal to zero, we obtain the condition of maximum I:

$$\ln\left[\frac{s}{\bar{N}} + e^{-s} - 1\right] = \frac{s}{\frac{e^s}{s+1} - 1}. \quad (69)$$

This equation may be solved by a numerical method for each specific value of $\bar{N}$. Figure 8 shows the dependence of the optimum amplitude of impulses received s_0 upon the mean number of impulses received in the time W^{-1}. If we know s_0, we may determine the maximum information efficiency of the recorder.

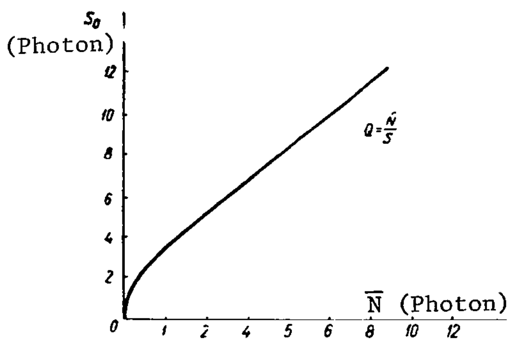

Figure 8

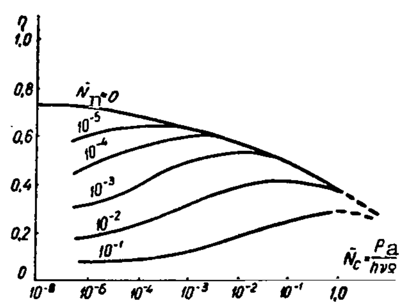

Figure 9

If there is additive noise in the channel, just as previously, we may compile a scheme of probabilities and may calculate the efficiency η. Figure 9 presents a graph showing the dependence of η on $\bar{N}_c$ for different mean numbers of photons of noise $\bar{N}_n$. In the limiting case $\bar{N}_c \ll 1$, $\bar{N}_n = 0$, we may obtain /312

$$I_{max} = \bar{N}_c \ln\frac{1}{\bar{N}_c}. \quad (70)$$

Comparing this expression with formula (14), we find that

$$\eta \approx \frac{1}{1 + \frac{\ln(1+\bar{N}_c)}{\bar{N}_c \ln\frac{1}{\bar{N}_c}}} \approx \frac{1}{1 + \frac{1}{\ln\frac{1}{\bar{N}_c}}} = 1 - \frac{1}{1 + \ln\frac{1}{\bar{N}_c}}. \quad (71)$$

The information efficiency of a binary recorder may be made arbitrarily close to unity by decreasing $\bar{N}_c$. However, as follows from Figure 9, even for very small $\bar{N}_c$, η differs to a considerable extent from unity. For large $\bar{N}_c$, a binary recorder is quite inefficient, as would be expected.

Quantum recorder in the case $P_a \gg h$. A wave guide maser with no noise may serve as an ideal energy-sensitive receiver for large population numbers of photons. This type of maser was studied in detail in (Ref. 12). Let us examine a segment of the wave guide which is filled by an active substance (Figure 10). We shall assume that the device operates at 0°k, and there is no radiation from the wave guide walls. Photons with an exponential distribution (13) enter the amplifier input. We must find the probability $p_n(k, L)$ of precisely k photons appearing at the output. This problem was completely solved in (Ref. 12). Based on the expressions obtained in this study for $p_n(k, L)$, we should be able to find the information efficiency of the amplifier for arbitrary amplification factors G. Unfortunately, attempts to perform this calculation have encountered computational difficulties. Therefore, we shall confine ourselves to the case $\bar{N}_c \gg 1$ and $G \gg 1$. For such an amplifier, we have /313

$$P_n(k) \equiv p_n(k, L) = \frac{\left(\frac{k}{G}\right)^n e^{-\frac{k}{G}}}{Gn!}, \tag{72}$$

i.e., the probability may be described by a Poisson distribution. The form of this distribution is shown in Figure 11.

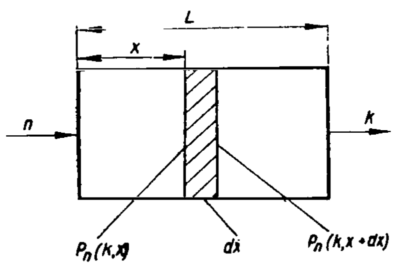

Figure 10

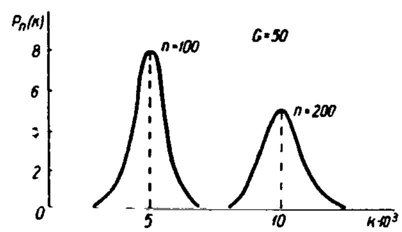

Figure 11

If there are n photons at the input, we may expect not only Gn photons at the output, but also a certain other number. Thus, a portion of the information is lost.

The output photon distribution may be described by the function

$$p(k) = \sum_n p(n) p_n(k), \tag{73}$$

where p (n) is determined according to (10) and (12).

After simple transformations, we obtain

$$p(k) = \alpha_k e^{\lambda_k k}, \tag{74}$$

where

$$\left.\begin{aligned} \alpha_k &= 1/(1+\bar{N})\,G \\ \lambda_k &= -\,1/(1+\bar{N})\,G \end{aligned}\right\}. \tag{75}$$

We shall be interested in the probability of recording n photons when k photons are transmitted. This probability equals

$$p_k(n) = p_n(k)\,\frac{p(n)}{p(k)}. \tag{76}$$

Since $p(n) = \alpha_n e^{\lambda_n n}$, then

$$p_k(n) = \left(\frac{ke^{\lambda_n}}{G}\right)^n \cdot \frac{1}{n!} \exp\left[-\frac{k}{G}(1+\lambda_k G)\right]. \tag{77}$$

This equation is valid, generally speaking, only in the case $0 \leq n \leq k$. However, we shall assume that it is satisfied for any n. The error of this approximation for large G and $\bar{N}$ is very small. Taking the fact into account that /314

$$1 + G\lambda_k = \frac{\bar{N}}{1+N} = e^{\lambda_n},$$

we may transform (77) to the following form

$$p_k(n) = \frac{M^n e^{-M}}{n!}, \tag{78}$$

where

$$M = \frac{ke^{\lambda_n}}{G}. \tag{79}$$

We have here obtained the probability of recording n photons in the case when k photons were transmitted. This probability may be described by the Poisson distribution with the average number of photons expected M. Let us now determine the arbitrary entropy. By definition, we have

$$H_k(n) = -\sum_{k=0}^{\infty} p(k) \sum_{n=0}^{\infty} p_k(n) \ln p_k(n). \tag{80}$$

Since

$$\ln p_k(n) = n \ln M - \\ - M - \ln(n!)$$

and it is assumed that n is sufficiently large, we may employ the Stirling formula

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln n - \\ - n + \ln\sqrt{2\pi}$$

Taking this relationship into account, we obtain

$$\ln p_k(n) = n \ln M - \\ -\left(n+\frac{1}{2}\right)\ln n - (M - \\ - n) - \ln\sqrt{2\pi}. \tag{81}$$

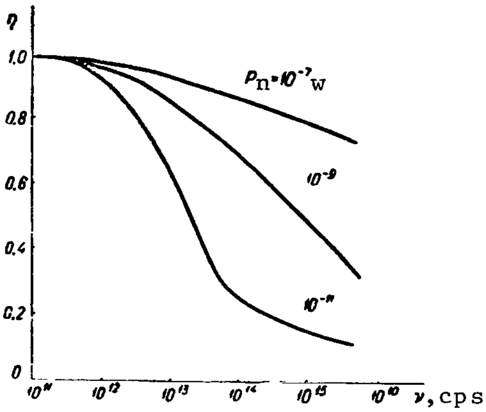

Figure 12

Substituting (81) in (80), we obtain

$$\sum_{n=0}^{\infty} p_k(n) \ln p_k(n) = \\ = M \ln M - \ln \sqrt{2\pi} - \sum_n p_k(n) \left(n + \frac{1}{2}\right) \ln n, \tag{82}$$

where the following equations are employed /315

$$\sum_{n=0}^{\infty} p_k(n) = 1;$$

$$\sum_{n=0}^{\infty} n p_k(n) = M.$$

Let us determine the last term in equation (89). For this purpose, let us expand ln n in power series with respect to $(n - M)/M$:

$$\ln n = \ln M + \ln\left[1 + \frac{n-M}{M}\right] = \\ = \ln M + \left[\frac{n-M}{M} - \frac{1}{2}\frac{(n-M)^2}{M^2} + \ldots\right]. \tag{83}$$

Substituting (83) in (82), and performing summation, we obtain

$$\sum_n p_k(n)\left(n + \frac{1}{2}\right) \ln n = \left(M + \frac{1}{2}\right) \ln M + \frac{1}{2} + 0\left(\frac{1}{M}\right). \tag{84}$$

The expressions for the moments of Poisson distribution are employed here. We finally have

$$-\sum_{n=0}^{\infty} p_k(n) \ln p_k(n) = \frac{1}{2} \ln (2\pi e M) + 0\left(\frac{1}{M}\right). \tag{85}$$

We may readily find the arbitrary entropy

$$H_k(n) = \frac{1}{2} \sum_k p(k) \ln (2\pi e M), \tag{86}$$

where M is a known function of k which is determined by relationship (79). We thus have

$$H_k(n) = \frac{1}{2} \sum_k \alpha_k e^{\lambda_k k} \ln\left[\frac{(2\pi e) e^{\lambda_n}}{G} k\right]. \tag{87}$$

In order to perform summation, we must take the fact into account that k is sufficiently large over the significant range within which it changes. Therefore, we may replace summation by integration with a high degree of accuracy:

$$H_k(n) \approx \frac{\alpha_k}{2} \int_0^{\infty} e^{\lambda_k k} \ln\left[\frac{(2\pi e) e^{\lambda_n}}{G} k\right] dk. \tag{88}$$

Since

$$\frac{1}{G} e^{\lambda_n} = \frac{\overline{N}}{(1 + \overline{N}) G} = -\overline{N} \lambda_k,$$

we have /316

$$H_k(n) = \frac{1}{2}\alpha_k \ln 2\pi e\overline{N} \int_0^\infty e^{\lambda_k k} dk + \frac{\alpha_k}{2} \int_0^\infty e^{\lambda_k k} \ln[-\lambda_k k]\, dk = $$
$$= \frac{1}{2} \ln 2\pi e\overline{N} + \frac{\alpha_k}{2} \int_0^\infty e^{-|\lambda_k| k} \ln(|\lambda_k| k)\, dk. \tag{89}$$

We have here taken the fact into account that $\lambda_k < 0$. Introducing the substitution $|\lambda_k|k = x$, we obtain

$$H_k(n) = \frac{1}{2} \ln 2\pi e\overline{N} + \frac{1}{2} \int_0^\infty e^{-x} \ln x\, dx. \tag{90}$$

The integral in the right hand side equals the Eüler constant

$$\int_0^\infty e^{-x} \ln x\, dx = C \approx 0{,}5777.$$

Thus, the final expression for the arbitrary entropy will be

$$H_k(n) = \frac{1}{2} \ln 2\pi e C\overline{N}. \tag{91}$$

The entropy H (n) was already computed [see formula (14)]. For large occupation numbers $\overline{N}$, it equals

$$H(n) \approx \ln(e\overline{N}). \tag{92}$$

Thus, the efficiency of the quantum recorder is

$$\eta = 1 + \frac{\ln 2\pi e C\overline{N}}{2 \ln e\overline{N}} = \frac{1}{2}\left(1 - \frac{\ln 2\pi C}{\ln e\overline{N}}\right). \tag{93}$$

For very large $\overline{N}$, we may disregard the second term in (93), and then the quantum recorder efficiency reaches its maximum value $\eta_{max} = \frac{1}{2}$. The loss of such a large amount of information may be explained by the fact that the described quantum recorder which is physically ideal does not extract information included in the signal phase. Actually, when $\overline{N}$ is sufficient large, a quantum description of the photon field approximates a classical description and, consequently, the phase information may comprise 50% of the total amount of information.

The fact that it is impossible to extract all the information of the signal by means of the quantum recorder does not mean that this same conclusion must be reached for any other receiver. Actually, although the quantum channel model which is employed assumes that a quantum recorder must be the optimum receiver, the model itself is not the only possible one. As was indicated above, the quantum channel may be described not only by the number of photons in each element $\Delta t \Delta \nu$, but also by any other selection of independent variables, provided that their total number is W. In particular, for large $\overline{N}$ a phase /317 description of the signal is possible. Therefore, we may expect that a phase-sensitive receiver will be more efficient under these conditions than a quantum recorder. This assumption will be substantiated below when a coherent amplifier is examined.

Coherent Amplifier

A quantum mechanical or parametric amplifier with a high amplification factor G may be employed as a coherent amplifier. It is valid to assume (Ref. 10, Ref. 11) that the noise at the amplifier output may be described by means of the effective noise at its input, which is expressed in terms of the voltage potential between the input signal and the noise, and increases classically [this is only valid for oscillations of zero field (Ref. 13)]. In the case of the quantum mechanical amplifier, the strength of the effective noise may be determined by the following expression

$$GP_{\text{eff}} = G\left[\frac{h\nu W}{\exp\left(\frac{h\nu}{kT_L}\right)-1} + \frac{h\nu W}{1+\exp\left(-\frac{h\nu}{k|T_M|}\right)}\right], \tag{94}$$

where T_L is the temperature of the attenuator and the amplifier wave guide; T_M -- negative temperature of the amplifying quantum ensemble. In an ideal temperature regime, $T_L = 0$, $|T_m| = 0$ and $P_{\text{eff. min.}} = h\nu W$. A similar expression may be obtained for a parametric amplifier. Since we have assumed that $G \gg 1$, consequently the strength of the output noise resulting from $P_{\text{eff.}}$ is always very much greater than the strength of the quantum noise $h\nu W$ In view of this fact, we may write the following with a high degree of accuracy

$$C_{\text{amp}} = W \ln\left(1 + \frac{P_a}{P_n + mh\nu W}\right), \tag{95}$$

where

$$m = \frac{1}{\exp\left(\frac{h\nu}{kT_L}\right)-1} + \frac{1}{1+\exp\left(-\frac{h\nu}{k|T_M|}\right)}. \tag{96}$$

Even for the optimum case, when m = 1, after coherent amplification the amount of information is decreased. Figure 9 shows the dependence of the amplifier efficiency

$$\eta = \frac{C_{\text{amp}}}{C_0}, \tag{97}$$

where $C_0 = W \ln (1 + P_a/P_n)$ for different values of the noise P_n ($W = 10^9$ cps, $T_n = 290°K$). For small values of $P_n \sim h\nu W$, the efficiency of the coherent amplifier rapidly decreases. As we have seen, under these conditions the quantum recorder is more efficient. If $P_n \gg h\nu W$, the coherent amplifier extracts practically all of the information included in the signal.

Conclusion /318

The basic assumptions of information theory change radically in the optical and infrared bands for a low level of eigen noise in the communication channel. The results presented above provide a comprehensive concept of the characteristics of the quantum channel. The general expression for the entropy capacity of narrow-band and wide-band communication systems was obtained, and the requirements for optimum signal characteristics were determined. The application of theory to an investigation of several receiving devices enabled us to clarify certain laws which are not described by classical theory. These laws were used to make recommendations regarding the use of a certain receiver in each specific case. We may sum up the results as follows.

1. A quantum narrow-band channel has a discrete photon distribution function, at which the signal entropy in the channel reaches a maximum. This function may be described by the exponential relationship

$$p(n) = \alpha e^{\lambda n},$$

where n = 0, 1, 2 ...

2. The signal entropy in a wide-band quantum channel reaches a maximum when the field of this signal coincides with the radiation field of an absolutely black body:

$$\bar{N}(\nu) = \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}.$$

If there is additive thermal noise in the channel, the information capacity is at a maximum when this requirement is imposed on the total signal-noise ensemble.

The entropy capacity is always finite, and is limited by the following value

$$H_{\max} = \pi\left[\frac{2P}{3h}\right]^{\frac{1}{2}}.$$

3. There are significant quantum phenomena in the case $h\nu W \sim P_n$. If $h\nu W \ll P_n$, we may completely change to the classical theory, as a result of which we obtain

$$H = \ln\frac{P_a}{P_n}$$

The only difference from the classical result is that now the absolute level from which the entropy is measured is established. This level is determined by the strength of the quantum noise.

4. A binary quantum recorder is the optimum recorder with respect to the information efficiency η for small signal intensities at the input of the receiving device, for which $h\nu W \leqq P_a$. If $P_a >> h\nu W$, then the ideal energy-sensitive receiver of the wave guide maser type cannot extract more than half of the information contained in the signal. Under these conditions, a coherent amplifier is the optimum device. The efficiency of this amplifier can be set very close to unity. /319

Kiev State University imeni T. G. Shevchenko

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